WNE Linear Algebra
Final Exam
Series A

1 February 2022

Questions

Please use a single file for all questions. Give reasons to your answers or provide a counterexample. Please provide the following data in the pdf file

- **name, surname and your student number,**
- **number of your group,**
- **number of the corresponding problem and the series.**

Each question is worth 4 marks.

Question 1.
Let $q\colon M(2\times 2;\mathbb{R})\to\mathbb{R}$ be a function given by the following formula

$$q(A)=\det A,$$

where $A\in M(2\times 2;\mathbb{R})$.

i) explain why q is a quadratic form on the space $M(2\times 2;\mathbb{R})$,
ii) find the matrix M of the form q relative to the basis

$$E_1=\begin{bmatrix}1&0\\0&0\end{bmatrix},\quad E_2=\begin{bmatrix}0&1\\0&0\end{bmatrix},\quad E_3=\begin{bmatrix}0&0\\1&0\end{bmatrix},\quad E_4=\begin{bmatrix}0&0\\0&1\end{bmatrix},$$

(that is, matrix $\begin{bmatrix}a&b\\c&d\end{bmatrix}$ is identified with the vector $(a,b,c,d)\in\mathbb{R}^4$),
iii) is the form q positive definite?

Solution 1. i)

$$\det\begin{bmatrix}a&b\\c&d\end{bmatrix}=ad-bc.$$

The function is homogeneous of degree 2.

ii)

$$M=\begin{bmatrix}0&0&0&\frac{1}{2}\\0&0&-\frac{1}{2}&0\\0&-\frac{1}{2}&0&0\\\frac{1}{2}&0&0&0\end{bmatrix}.$$

iii) no, it is not,

$$\det\begin{bmatrix}1&0\\0&-1\end{bmatrix}=-1<0.$$

Question 2.
Let $P \in M(n \times n; \mathbb{R})$ be a matrix of an orthogonal projection. Is

$$Q = I - P,$$

a matrix of some orthogonal projection? Note that

$$I = I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix},$$

it the $n \times n$ unit matrix.

Solution 2.
Yes, if P is the matrix of orthogonal projection onto subspace $V \subset \mathbb{R}^n$, then $P - I$ is the matrix of orthogonal projection onto $V^{\perp}$.

Question 3.
Do there exist two subspaces $V, W \subset \mathbb{R}^3$ such that $\dim V = \dim W = 2$ and

$$V \cap W = \{\mathbf{0}\},$$

(i.e. the common part is the zero vector)? Give an example or explain why it is not possible.

Solution 3.
No, they do not. Any subspace of dimension 2 in $\mathbb{R}^n$ is given by a single, non-zero linear equations. Matrix of a system of two non-zero linear equations has rank either 1 or 2. Therefore, the dimension of the set of solutions is either $3 - 1 = 2$ or $3 - 2 = 1$.

Question 4.
Let $A \in M(8 \times 8; \mathbb{R})$ be a block matrix, with each block B_{ij} of size 2×2, i.e.

$$A = \left[\begin{array}{c|c|c|c} B_{11} & B_{12} & B_{13} & B_{14} \\ \hline \mathbf{0} & B_{22} & B_{23} & B_{24} \\ \hline \mathbf{0} & \mathbf{0} & B_{33} & B_{34} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & B_{44} \end{array}\right],$$

where

$$B_{11} = \begin{bmatrix} 2022 & 2021 \\ 2021 & 2022 \end{bmatrix}.$$

Is $\lambda = 1$ an eigenvalue of matrix A?

Solution 4.
Yes, it is.

$$\det(B_{11} - I) = \det \begin{bmatrix} 2021 & 2021 \\ 2021 & 2021 \end{bmatrix} = 0,$$

and the determinant of the upper diagonal block matrix is the product of the matrices on the diagonal, i.e.

$$w_A(1) = \det(A - I) = 0.$$

Question 5.
For any $\alpha, \beta \in \mathbb{R}$ let $F(\alpha, \beta) \in M(3 \times 3; \mathbb{R})$ be a matrix such that

$$F(\alpha, \beta) = \begin{bmatrix} 1 & \alpha & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Does it follow that for any $\alpha, \beta \in \mathbb{R}$

$$(F(\alpha, \beta))^{-1} = \begin{bmatrix} 1 & -\alpha & -\beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}?$$

Solution 5.
Yes, it does. It can be check by direct computation that (it is enough to check the first product only)

$$\begin{bmatrix} 1 & \alpha & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\alpha & -\beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\alpha & -\beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$